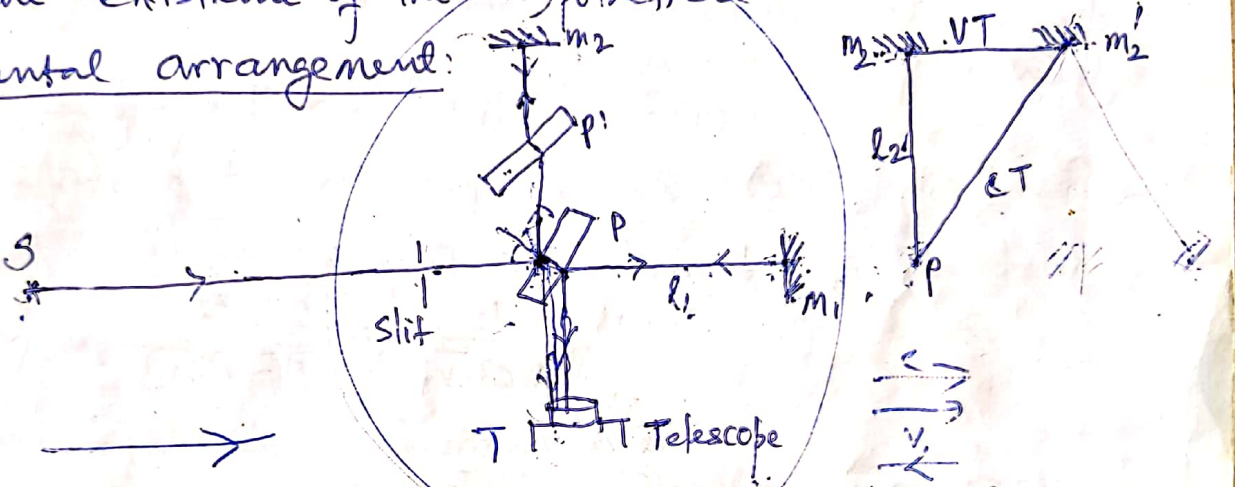
MICHELSON-MORLEY EXPT.

Galilean transformation equations are valid in inertial frames of reference. However no frame on earth is inertial due to rotating motion of earth. In search of such a frame Scientists (Maxwell & others) presumed the existence of a hypothetical medium pervading all space called ether. To establish its existence and relative speed of light wrt. this medium a series of expt. were performed. If 'c' is velocity of light in ether, its velocity in any other inertial frame moving with velocity 'v' wrt. to ether shall be $c' = c \pm v$. Michelson & Morley's precession expt is a mile stone in this regard. Using Michelson interferometer results of the experiment with incredible sensitivity and accuracy could'nt establish the relative velocity of earth wrt ether and hence the existence of the hypothetical medium itself.

Experimental arrangement:



The experimental setup has been shown in the diagram. A monochromatic beam of light from a source 'S' after passing through a narrow slit is allowed to fall on a glass plate at 45° whose front surface is silvered. The plate splits the beam into two parts. The transmitted ray ~~strikes~~ passes through the plate to strike the mirror M_1 from where it gets reflected back to P and enters the telescope T. The second part after reflection at P' travels to mirror M_2, gets reflected & then passes through P to enter the telescope T. A compensating plate P' of same thickness as P is placed on the path of the second ray so that the optical path travelled by both the rays are same before entering the telescope T. The two rays superpose inside the telescope and produce interference fringes.

Theory: The two rays travel diff. optical path $l_1 = PM_1$ and $l_2 = PM_2$ (to and fro). If a bright fringe is seen on the crosswire of T, the path difference between the two rays must be an integral multiple of λ i.e.

$$2(l_1 - l_2) = n\lambda \quad ; \quad \text{---} (1)$$

where 'n' is an integer.

Now suppose that the entire appts is kept in the lab. in such a way that path PM_1 is parallel to the direction of motion of the earth through ether. That is the appts is moving at a speed v in direction PM_1 relative to ether, where v is the velocity of earth. So the time taken by the light ray from P to M_1 and back is

$$t_1 = \frac{l_1}{c-v} + \frac{l_1}{c+v} = \frac{2l_1 c}{c^2-v^2} = \frac{2l_1/c}{1-v^2/c^2} \quad \ldots (2)$$

Here $c' = c-v$ is the velocity of light relative to M_1 when light moves towards M_1 & $c' = c+v$, when light moves back from M_1.

In order to calculate the time taken by the second light ray, it may be noted that M_2 moves towards in the direction PM_1 with velocity v, as light starting from P travels towards M_2. If light takes a time T to reach M_2, then as shown in diagram,

$$(cT)^2 = l_2^2 + (vT)^2 \Rightarrow T = \frac{l_2}{\sqrt{c^2-v^2}}$$

Similarly time taken during return to P is also $T' = \frac{l_2}{\sqrt{c^2-v^2}}$

So total time taken by the second ray is

$$t_2 = T + T' = \frac{2l_2}{\sqrt{c^2-v^2}} = \frac{2l_2/c}{\sqrt{(1-v^2/c^2)}} \quad \ldots (3)$$

The two rays travel through different times and consequently interefer with a phase diff.

$$\phi = \omega \Delta t = \omega (t_1 - t_2) = \frac{2\pi c}{\lambda} (t_1 - t_2) \quad \ldots (4)$$

If the apparatus is now rotated by 90°, the arm PM_2 points along the direction of motion of earth and PM_1 lies perpendicular to it. The phase difference of the two interfering rays now becomes

$$\phi' = \omega \Delta t' = \omega (t_1' - t_2') = \frac{2\pi c}{\lambda} (t_1' - t_2') \quad \ldots (5)$$

where $t_1' = \frac{2l_2/c}{1-v^2/c^2}$ $\ldots 6(a)$, $t_2' = \frac{2l_1/c}{\sqrt{1-v^2/c^2}}$ $\ldots 6(b)$

The change of phase from $\phi \to \phi'$ must lead to a shift in the interference pattern. As we know there shall be a shift of 1 fringe through the crosshire for a change of phase by a factor 2π. Thus the expected no. of fringe shift for a total phase change of $(\phi + \phi')$ through the crosshire of the telescope shall be given by as detailed below.

$$\Delta m = \frac{\phi_1 + \phi_2}{2\pi}$$

$$= \frac{2\pi c}{\lambda} \cdot \frac{1}{2\pi} \left[(t_1 - t_2) + (t_1' - t_2') \right]$$

$$= \frac{2\pi c}{2\pi\lambda} \left[\frac{2l_1/c}{1 - v^2/c^2} - \frac{2l_2/c}{\sqrt{1 - v^2/c^2}} + \frac{2l_2/c}{1 - v^2/c^2} - \frac{2l_1/c}{\sqrt{1 - v^2/c^2}} \right]$$

$$= \frac{2c}{\lambda} \left[\frac{2(l_1 + l_2)}{c} \right] \left[\frac{1}{1 - v^2/c^2} - \frac{1}{(1 - v^2/c^2)^{1/2}} \right]$$

$$= \frac{2(l_1 + l_2)}{\lambda} \left[\frac{c^2}{c^2 - v^2} - \left(\frac{1}{1 + v^2/2c^2} \right) \right]$$

$$= \frac{2(l_1 + l_2)}{\lambda} \left[\frac{1 - (1 - v^2/c^2)^{1/2}}{1 - v^2/c^2} \right]$$

$$= \frac{2(l_1 + l_2)}{\lambda} \left(\frac{1 - 1 + \frac{1}{2} v^2/c^2}{1 - v^2/c^2} \right) \simeq \frac{(l_1 + l_2)}{\lambda} v^2/c^2 \quad \text{(higher order terms are neglected)} \quad \text{---(7)}$$

(expanding binomially & with $v \ll c$)

$c = 299792458$ mt/sec

$* \; (1 - v^2/c^2)^{1/2} = 1 - \frac{1}{2} v^2/c^2$ neglecting higher terms

Michelson and Morley in their expt took

$l_1 = l_2 = 11$ mtrs, $\lambda = 5.6 \times 10^{-7}$ mtr, $V = 3 \times 10^4$ m/sec

$c = 3 \times 10^8$ mt/sec, $(i.e.\ v/c = 10^{-4})$,

$\Delta m = 0.4$ fringes. ——(8)

This represent the expected fringe shift across the cross wire of the apparatus; if at all there exist a relative velocity of earth with respect to the hypothetical ether medium.

The experiment was performed along and right at different places on the globe and at different times in a year. The appts used was sensitive enough to measure shift in one hundredth of a fringe. Inspite of all care taken and experiment performed with utmost precission and accuracy, viz. such fringe shift was observed. Even though the expt was repeated by other group of scientists, it also did not yield any fringe shift. Hence it is evident from (7) that for $\Delta m = 0$, $V = 0$. Thus there is no relative velocity of earth with respect to ether.

The null result of Michelson-Morley expt was totally unexpected according to Galilean relativity & Maxwell's theory of light. Several attempts were made to account for it, but without much success.

Postulates of Special Theory of Relativity: Einstein's Idea:

Several attempts were made to explain the negative results of the carefully performed experiment by Michelson-Morley & other group of Scientist; but the motion of earth through stationary ether could not be detected. Hence "absolute motion through ether" appeared to be a totally wrong concept.

Einstein started to tackle the problem with a radically new idea that, "motion through ether is meaningless concept, while motion relative to material bodies alone has physical significance". This led to the rejection of the old idea of not only absolute space but also of absolute time.

Einstein relativity consist of two parts:

1. Special or Restricted relativity → which deals with problems in Uniform rectilinear motion
2. General relativity → which deals with non uniform motion.

Here we shall deal with special theory of relativity and its consequences. The formulations are based on two important postulates known as Postulates of Special Relativity

A. Principle of Equivalence: The Laws of physical phenomena are same when stated in terms of two systems of reference in Uniform translatory motion relative to each other.

B: Principle of Constancy of velocity of light: The velocity of light in vaccume is constant, independent not only of direction of propagation but also of relative velocity of the source and observer.

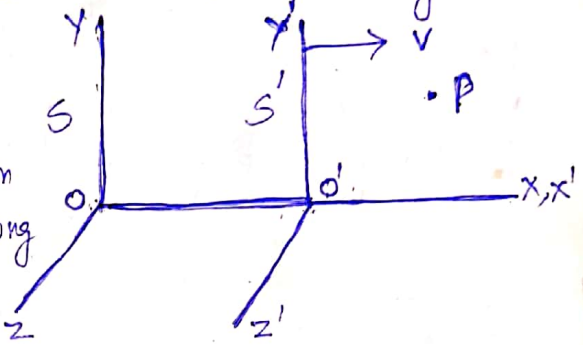
The first postulate implies that not only laws of mechanics, but also that of electrodynamics and optics should be same for all inertial frames of reference. It also asserts that all inertial frames are equivalent and there is no place for a privileged frame like Absolute Space or ether.

The second postulate affirms the constancy of velocity of light and its independence from the relative motion between the source and observer. Even though it can't be verified by any direct expt. the interpretation of the -ve result of experiments performed by Michelson-Morley, Romer et all stands in support of the idea. This principle led to important conclusions like futility of ether, intrinsic quantum nature of radiation, variation of mass with vel & mass energy equivalence.

The Lorentz transformation

Lorentz Transformation Equations:

These transformations relate the space-time co-ordinates of an event as observed from two inertial frames of refer. in relative motion with respect to one another. These are based on the postulates of special relativity and leads to throw light on many interesting consequences such as simultaneity of space-time, length contraction, time dilation, velocity transformation etc.



Consider two inertial frames of reference S and S'. The frame S' have a uniform translatory velocity v relative to S along x direction. We choose x and x' axes of the two frames to always coincide. This implies y-z plane to move parallel to y-z plane.

Let an event P occur at any space-time position such that, the co-ordinates of the event as observed from S and S' frames are (x, y, z, t) and (x', y', z', t') respectively (say). Since space time is uniform, the relation between the two set of co-ordinates must be linear. So the transformation equation can be written in a generalised form

$$x' = a_{xx}x + a_{xy}y + a_{xz}z + a_{xt}t + b_1 \quad \text{---(1a)}$$
$$y' = a_{yx}x + a_{yy}y + a_{yz}z + a_{yt}t + b_2 \quad \text{---(1b)}$$
$$z' = a_{zx}x + a_{zy}y + a_{zz}z + a_{zt}t + b_3 \quad \text{---(1c)}$$
$$t' = a_{tx}x + a_{ty}y + a_{tz}z + a_{tt}t + b_4 \quad \text{---(1d)}$$

Here $a_{xx}, \ldots b_4$ are constants, which are independent of x, y, z, t.

Our initial conditions are such that, the origins of the two co-ordinate systems coincide initially when $t = t' = 0$ i.e. $x=0, y=0, z=0$ implies $x'=0, y'=0, z'=0$. Using these conditions in eqn 1(a) — 1(d) we get

$$b_1 = b_2 = b_3 = b_4 = 0 \quad \text{——(2)}$$

Such transformation with all b's zero are called homogeneous transformation. Otherwise they are called inhomogeneous. Here we shall deal with homogeneous transformation only, without loss of any physical consequence.

The x'-z' plane has been so choosen that, it always coincide with x-z plane. Hence if $y=0$, then $y'=0$. Using

3(a)

this condition in 1(b)

$$0 = a_{yx}x + a_{yz}z + a_{yt}t$$

Here x, z and t are linearly independent. Hence for the above equation to hold true we must have

$$a_{yx} = a_{yz} = a_{yt} = 0 \quad \text{—— 3(b).}$$

Using cond^n 3(a) and 3(b) in I(b) we get

$$y' = a_{yy}y \quad \text{—— 3(c)}$$

The constant a_{yy} may depend on the relative velocity $\vec{V}$ of S' wrt. S. According to Einsteins postulate of equivalence all inertial frames are equivalent. Hence one may assume S' to be stationary and S moving relative to S' in opposite dir^n with a velocity v. This then leads to relation

$$y = a_{yy}y' \quad \text{—— 3(d)}$$

Finally multiplying 3(c) and 3(d) we get

$$yy' = a_{yy}^2 yy' \Rightarrow a_{yy}^2 = 1 \text{ or } a_{yy} = \pm 1 \quad \text{—— 3(e)}$$

The proper solution of 3(e) is $a_{yy} = +1$, since we expect that by continuously decreasing $\vec{V}$ to zero, we should get $y' = y$ (identity transformation). Other wise it may lead to space inversion with $y' = -y$). Thus for the proper transformation

$$y' = y \quad \text{——} \quad (A)$$

Similar arguments hold true for z co-ordinate and hence.

$$z' = z \quad \text{——} \quad (B)$$

Let us look for transformation of x co-ordinate.

$$x' = a_{xx}x + a_{xy}y + a_{xz}z + a_{xt}t \quad \cdots\cdots (4)$$

If we consider a particle to be at rest in frame S' at pos^n $x' = 0, y', z'$; when observed from S, this particle moves with a velocity $\vec{V}$ along x axis. Thus the co-ordinates of the particle viewed from S frame are $x = Vt, y, z$ at any time t. Hence we have

$$0 = a_{xx}Vt + a_{xy}y + a_{xz}z + a_{xt}t$$

$$0 = (Va_{xx} + a_{xt})t + a_{xy}y + a_{xz}z$$

As the transformations are linear the coeff. of t, y, z should vanish i.e. $a_{xy} = 0$, $a_{xz} = 0$ and $a_{xt} = -Va_{xx}$

Using these in (4) we get

$$x' = a_{xx}x + 0 + 0 - va_{xx}t$$

$$x' = a_{xx}(x - vt) \quad \text{..... } 4(a)$$

However a_{xx} depends only on the relative velocity between S and S'. If S' is assumed stationary and S moving with vel $-v$ along x' axis, we have by symmetry

$$x = a_{xx}(x' + vt') \quad \text{..... } 4(b)$$

According to the principle of constancy of light, the vel. of light is same in all directions & independent of rel. motion of the inertial frames. Initially at $t = t' = 0$, when the origins of the two frames coincide, let a light signal originate from common origin. The signal arrives at a point on x axis, whose coordinates when viewed from S and S' frame respectively are (x, t) & (x', t'). Hence we get

$$x = ct \quad \text{and} \quad x' = ct' \quad \text{——4(c)}$$

Substituting values in 4(c) in 4(a) and 4(b),

$$ct' = a_{xx}(ct - vt) = a_{xx}(c - v)t \quad \text{——4(d)}$$

$$ct = a_{xx}(ct' + vt') = a_{xx}(c + v)t' \quad \text{4(e).}$$

multiplying 4(d) and 4(e) we get

$$c^2tt' = a_{xx}^2(c^2 - v^2)tt'$$

$$\Rightarrow a_{xx} = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{..... (5)}$$

Substituting this value in 4(a) & 4(b) we get

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad \text{..... (C).}$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{..... (C')}$$

eliminating x' from C & C', we get

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{——(D)}$$

The complete set (A), (B), (C), (D) are the proper L.T. Eqn

i.e. $x' = \frac{(x - vt)}{\sqrt{1 - v^2/c^2}}$, $y' = y$, $z' = z$, $t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$

Physical consequences of Lorentz transformation:

1. Relativity of Simultaneity:

Two events occurring at two space positions with co-ordinates x_1 and x_2 in an inertial frame S, are said to be simultaneous, if they occur at the same time t. This time is noted by two synchronised clocks placed at x_1 and x_2 in 'S'. The same event is observed from another moving Lorentz frame S' at (x_1', t_1') and (x_2', t_2') respectively. We have

$$x_1' = (x_1 - vt)\gamma, \quad t_1' = \left(t - vx_1/c^2\right)\gamma$$

$$x_2' = (x_2 - vt)\gamma, \quad t_2' = \left(t - vx_2/c^2\right)\gamma$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

From the above Lorentz transformation equations we observe that, the event do not occur in S' frame simultaneously i.e. $t_1' \neq t_2'$. The interval difference of the occurrance of the event in S' frame is given by

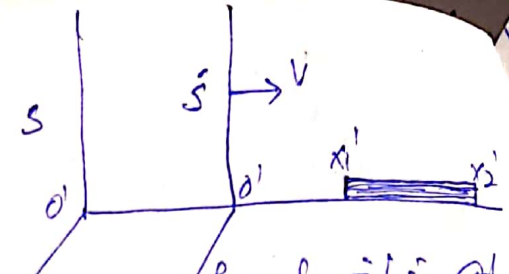
$$t_2' - t_1' = \frac{\gamma v}{c^2}(x_1 - x_2)$$

Thus the simultaneity of an event in S is not simultaneous in the frame S' but relative to the frame of reference. Thus two events that occur simultaneously in one frame is not simultaneous in another frame.

2. Length Contraction:

The measurement of length involves the concept of rod scale. The positions of both ends of the rod must be measured simultaneously on a frame, in order to define its length.

Consider two frames S and S'. S' moving relative to S with a velocity 'v' along 'x' axis. Suppose the rod is at rest in frame S' with its length along x axis. The rod, thus is a moving object wrt frame 'S' (since S' is moving wrt S with vel. v).



let x_1' and x_2' are co-ordinates of the rod with respect to O'.
Then $l_0 = x_2' - x_1'$ .....(i)
represent length of the rod in S', in which it is at rest.
So S' is called rest or proper frame of rod and l_0 is called rest length or proper length of the rod.

The co-ordinates of this moving rod when measured in frame S are x_1 and x_2. But the distance $x_2 - x_1$ represent the length of the rod if the co-ordinates x_1 & x_2 have been measured at the same time $t_1 = t_2 = t$ in frame S. Now the co-ordinates (x_1, t) and (x_2, t) in S' are related to their counter part in S' by the relation

$$x_1' = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \quad x_2' = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} - \frac{v(t_2 - t_1)}{\sqrt{1 - v^2/c^2}} \to 0$$

$$x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} \quad \text{since } t_1 = t_2 = t.$$

$$\text{or} \quad l_0 = \frac{l}{\sqrt{1 - v^2/c^2}} = \gamma l \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\text{or} \quad l = l_0/\gamma = l_0\sqrt{1 - v^2/c^2}$$

Since $\gamma < 1$, $l < l_0$.

Thus the length of the moving rod in the direction of its motion is observed to be shorter than its proper length. However there is no change in the dimension of the rod along 1^{r} dirn i.e. y & z directions.

Ex.. If a ball moves at rel comparable to light then it shall appear as a flat disk along its direction of motion.

Relativity of Time ~ time dialation:

The most remarkable consequence of the transformation equations is the fact that $t \neq t'$. Ordinarily this means that the clocks in the two frames S & S', which are in relative motion, run at different rates. Let $\Delta t = t_2 - t_1$ be an interval measured by a clock placed at origin of frame S. The same interval measured from another synchronized clock placed at origin of S' (which is in relative motion w.r.t S) is $\Delta t' = t_2' - t_1'$. Let us find the relationship between Δt and $\Delta t'$.

$$\Delta t' = t_2' - t_1'$$

$$= \gamma\left(t_2 - vx_2/c^2\right) - \gamma\left(t_1 - vx_1/c^2\right) \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad x_2 = x_1 = x' \text{ at origin}$$

$$\simeq \gamma(t_2 - t_1) = \gamma \Delta t = \frac{\Delta t}{\sqrt{1 - v^2/c^2}}$$

Since $v << c$, ~~$\Delta t' < \Delta t$~~ $\Delta t' > \Delta t$

Thus there are two proper times; one for the observer at rest (in S) and the other for the observer in motion (in S'). The time is greater for the observer at rest than the observer in motion. In other words the clock with the observer in motion runs slower than the observer at rest (time is dilated)

Moreover 'γ' involves v^2. So the above conclusion is independent of sign. In other words the opinion is reciprocal. Hence to the observer in S', the clock in S shall appear to run slow (since S is moving with a vel. $-v$ w.r.t. S'). The general conclusion is that both the observer will feel that the other's clock is running slower and hence lands in a "clock paradox". The only acceptable conclusion is that, the moving clock runs slower

Relativity of Simultaneity:

Consider two events happening in system S fixed by (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2). Their time of occurance in frame S' is given by the transformation equation

$$t_1' = \gamma(t_1 - vx_1/c^2), \quad t_2' = \gamma(t_2 - vx_2/c^2).$$

Suppose that the two events are simultaneous in S, i.e. $t_1 = t_2$, but $x_1 \neq x_2$. Hence $t_1' \neq t_2'$ which means that the same two events are not simultaneous when observed from system S'. The concept of simultaneity, unlike that of time is also relative but not absolute.

Addition of velocities:

The transformation equations lead to a relativistic formula for addition of velocities, that are different from classical mechanics.

Let us consider the motion of an object viewed from two frames S & S' in relative motion with vel 'v'. Let u and u' be velocity of the object measured from S and S'. If dx is the distance moved in time dt' in S. The same is dx' in time dt' in S' such that (motion is confined to x dir^n alone).

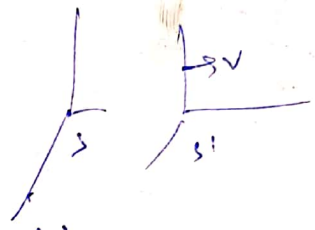
$$u = \frac{dx}{dt}, \quad u' = \frac{dx'}{dt'}$$

We have the transformation Eq^n

$$x = \gamma (x' + vt')$$

$$t = \gamma \left(t' + vx'/c^2\right)$$

$\gamma : \frac{1}{\sqrt{1 - v^2/c^2}}$



differentiating

$$dx = \gamma (dx' + v\,dt')$$

$$dt = \gamma \left(dt' + v\frac{dx'}{c^2}\right)$$

$$\therefore \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + v\,dx'/c^2} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} = \frac{u' + v}{1 + vu'/c^2}$$

$$\Rightarrow \boxed{u = \frac{u' + v}{1 + vu'/c^2}}$$

This is the relativistic law for addition for velocities and is different from the classical law ($u = u' + v$).

The following Conclusions are drawn:

i) On the non-relativistic limit i.e. when $v \ll c$ or $u' \ll c$

$u = u' + v$ or $u' = u - v$ is the Galilean transformation eq^n.

ii) If $u' = c$ then $u = \frac{c + v}{1 + cv/c^2} = c$

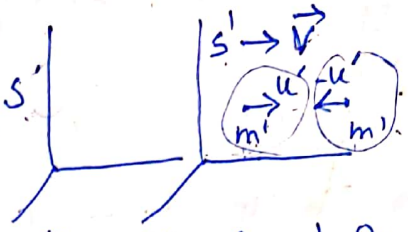
This means that if the velocity of the body u' as measured in S' is the velocity of light c, the velocity of the same particle as measured in S is also c. Thus velocity of light is an absolute constant, independent of velocity of system of reference.

iii) If $v = c$, then $u = c$. i.e. what ever be the velocity that is combined with 'c' the resultant is always c. So the velocity of light is the ultimate attainable limit.

'c' enjoys a position in relativity, which is akin to "∞" in mathematics

Variation of mass with Velocity:

According to classical electrodynamics, mass of a body is constant and is independent of velocity. But according to theory of relativity, mass varies with velocity as explained and derived below



Consider two bodies of mass m' are in System S'. They are moving parallel to 'X' axis with vel u' & $-u'$. They collide & coalesce to form a single body after collision.
Applying the principle of conservation of momentum, the two bodies shall be at rest after collision inside the System S' and have a mass $2m'$. But this collision does not appear to be so when viewed from the System S.

The velocities u' and $-u'$ transform according to formula

$$u_1 = \frac{u'+v}{1+u'v/c^2} \quad \text{and} \quad u_2 = \frac{-u'+v}{1-u'v/c^2} \qquad \cdots\cdots (1)$$

where u_1 and u_2 are velocities of the two bodies when viewed from System S and m_1 & m_2 are their masses (say).
Since the bodies are at rest in S' after collision, they appear to travel with vel. v with respect to S. Applying principle of conservation of momentum ~~in Systems, after~~, which holds true in all frames

$$m_1u_1 + m_2u_2 = (m_1+m_2)\,v \qquad \text{——(2)}$$

Substituting the values of u_1 and u_2 from equation (1) in (2) we get

$$\frac{m_1}{m_2} = \frac{1+u'v/c^2}{1-u'v/c^2} \qquad \text{——(3)}$$

using eqn (1) in (3) we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1-u_2^2/c^2}}{\sqrt{1-u_1^2/c^2}} \qquad \text{——(4)}$$

If the mass m_2 is moving with zero velocity before collision in System S $u_2 = 0$, (4) gives us

$$m_1 = \frac{m_2}{\sqrt{1-u_1^2/c^2}}$$

Now both the bodies have the same mass when moving with the same velocity, the above equation means that m_1 is the mass of the body when it is moving with velocity u_1 and m_2 is the mass when it is at rest.

So in commonly used notation $m_1 = m$, $m_2 = m_0$ and $u_1 = v$. So that

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad \text{——— (5)}$$

This is the relativistic formula for variation of mass with velocity. m_0 is called rest mass and m is called effective mass of the body in motion.

In this limit $v \to c$, $m \to \infty$ which suggest that material bodies can't attain speed of light.

Mass ~ Energy Relation:

The Eqn of motion in relativistic mechanics is given by

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{d}{dt}\left(\frac{m_0\vec{v}}{\sqrt{1 - v^2/c^2}}\right)$$

The concept of work done by a force and that of K.E/P.E is equally valid in relativistic mechanics as well. According to work-energy theorem, "The work done by a force on a body is equal to change in its K.E"; which implies

$$dE_K = F\,dx = \frac{dp}{dt}\cdot dx = \frac{d}{dt}(mv)\,dx = v\,d(mv) = v^2 dm + mv\,dv \quad \text{——— (1)}$$

where $m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \Rightarrow m^2 = \frac{m_0^2 c^2}{c^2 - v^2}$ (On squaring)

$$\Rightarrow m^2c^2 - m^2v^2 = m_0^2c^2$$

$$\Rightarrow c^2 2m\,dm - v^2 2m\,dm - m^2 2v\,dv = 0 \quad \text{(On differentiation)}$$

($m_0, 2, c$ are const).

$$\Rightarrow c^2 dm - v^2 dm - mv\,dv = 0$$

$$\Rightarrow mv\,dv + v^2 dm = c^2 dm \quad \text{——— (2)}$$

From (1) & (2) $\quad dE_K = c^2 dm \quad \text{---- (3)}$

This suggests that the change in kinetic energy is equal to change in mass (dm) times c^2. As a particle starts from rest, its mass increases from $m_0 \to m = m_0/\sqrt{1 - v^2/c^2}$, the kinetic energy acquired is given by integrating (3) i.e. $\int_0^{E_K} dE_K = \int_{m_0}^{m} c^2\,dm$

$$\therefore \boxed{E_K = c^2(m - m_0) = mc^2 - m_0c^2} \quad \text{——— (4)}$$

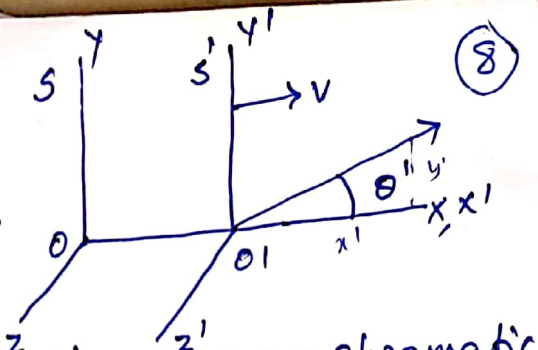
The above equation gives relativistic K.E of the particle. The term m_0c^2 is called rest mass energy - it is the energy stored in the form of mass when the body is at rest. As the particle acquires velocity, its K.E is given by (4). The total energy is given by $E = E_K + m_0c^2 = mc^2$ ——— (5)

In non relativistic limit ($v \ll c$), K.E is given by

$$E_K = mc^2 - m_0c^2 = \frac{m_0}{\sqrt{1 - v^2/c^2}}c^2 - m_0c^2 = m_0c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) = m_0c^2\left[\left(1 + \tfrac{1}{2}v^2/c^2\right) - 1\right] = \frac{1}{2}m_0v^2$$

Eqn (5) is a verification of principle of conservation of mass & energy to mass-energy equivalence.

Doppler Effect in Relativity:



Consider a plane electromagnetic wave propagating in space. S' has origin at s', amplitude unity (1), monochromatic, and rays travelling at angle θ' with x axis.

Such a wave is represented by the expression when viewed from S' frame as:

$$\gamma = \cos 2\pi \left[\frac{x' \cos\theta' + y' \sin\theta'}{\lambda'} - \nu' t' \right] \quad \text{---(1)}$$

When viewed from S frame it appear as

$$\gamma = \cos 2\pi \left[\frac{x \cos\theta + y \sin\theta}{\lambda} - \nu t \right] \quad \text{---(2)}$$

Obviously the velocity in S' frame is ν' and in S frame ν. and are same as the wave is a electromagnetic one.

Let us apply Lorentz transformation to eq^n (1) namely $x' = \frac{x - Vt}{\sqrt{1-\beta^2}}$ and $t' = \frac{t - \frac{V}{c^2}x}{\sqrt{1-\beta^2}}$ (where $\beta = V/c$)

Now eq^n (1) becomes

$$\cos 2\pi \left[\frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}} x + \frac{\sin\theta'}{\lambda'} y - \frac{(\beta\cos\theta' + 1)\nu'}{\sqrt{1-\beta^2}} t \right] \quad \text{---(3)}$$

Eq^n (3) represent the same electromagnetic wave in S frame and should be identical with (2). Hence comparing the co-eff. of x, y, t from (2) & (3) (in rhs) we get

$$\frac{\cos\theta}{\lambda} = \frac{\cos\theta' + \beta}{\lambda'\sqrt{(1-\beta^2)}} \quad \text{--- 4(a)}$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'} \quad \text{--- 4(b)}$$

$$\therefore \nu = \frac{\nu'(1 + \beta\cos\theta')}{\sqrt{1-\beta^2}} \quad \text{--- 4(c)}$$

Of course we always have the known condition $\lambda'\nu' = \lambda\nu = c$ --- 4(d)

Eqn 4(c) gives us an expression for the frequency of the spectrum ν (as observed from S frame) in terms of ν' (the frequency in S' frame)

$$\nu = \nu' \frac{(1+\beta\cos\theta')}{\sqrt{1-\beta^2}} \quad \text{or} \quad \nu' = \nu \frac{(1-\beta\cos\theta)}{\sqrt{1-\beta^2}} \quad \text{---5(a)}$$

In the classical limit $v << c$, we neglect the the higher order terms in Binomial expansion of $(1-\beta^2)^{-1/2}$ and retaining only terms upto 1st order

get from 5(b) $\nu = \frac{\nu'(1-\beta^2)^{1/2}}{1-\beta\cos\theta} \simeq \frac{\nu'}{1-\beta\cos\theta} \simeq \nu'(1-\beta\cos\theta)^{-1} \simeq \nu'(1+\beta\cos\theta)$

This is the classical result:

with $\theta = 0$ → observer is seeing the source moving towards him

$$\nu = \nu'(1+\beta) = \nu'\left(1+\frac{v}{c}\right) \quad \text{---6(a)}$$

⇒ observed frequency $\nu >$ proper freq ν'

with $\theta \Rightarrow 180$ → Source moving away from him

$$\nu = \nu'(1-\beta) = \nu'(1-v/c) \quad \text{---6(b)}$$

⇒ observed frequency $\nu < \nu'$ (proper freq)

and with $\theta = 90^\circ$, there is no Doppler Effect ($\nu = \nu'$)

In the relativistic limit: When v is comparable to c, we have to retain second order terms in the expansion of $\sqrt{1-\beta^2}$.

For longitudinal Case i.e. either $\theta = 0^\circ$ or 180° we have the following results

with $\theta = 0^\circ$ from 5(b): $\nu = \frac{\nu'(1-\beta^2)^{1/2}}{1-\beta\cos\theta} = \nu'\frac{\sqrt{(1-\beta^2)}}{1-\beta} = \nu'\sqrt{\frac{(1-\beta^2)}{(1-\beta)^2}}$

$$= \nu'\sqrt{\frac{1+\beta}{1-\beta}} = \nu'\sqrt{\frac{c+v}{c-v}} \quad \text{---7(a)}$$

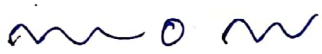
With $\theta = 180^\circ$ $\quad \nu = \nu' \sqrt{\frac{c-v}{c+v}}$ — 7(b)

7(a) and 7(b) represent Longitudinal Doppler shift. and was experimentally confirmed by in the Laboratory.

With $\theta = 90^\circ \rightarrow$ The effect is Transverse Doppler effect (which is not seen in classical limits), is an purely relativistic one. We get from 5(b)

$$\nu = \nu' \sqrt{1-\beta^2} \quad \text{— 7(c)}$$

This shows that if the line of sight is 90° to the relative motion of the source & observer the frequency ν is less than the proper freq ν' ($\nu < \nu'$). These results have been experimentally Confirmed.



MINKOWSKI SPACE ~ 4-D formulation

The Lorentz transformation equations, that connects the space time co-ordinates of an event in two inertial frames S & S' in relative motion, are consequences of the postulates of special theory of relativity. These equations are

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z \quad \& \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad \text{---(1)}$$

From the 1st and 4th equation it is clear that, the space co-ordinate depend on time and the time co-ordinate on the space co-ordinate. It is therefore quite natural not to treat the two co-ordinates separately, but together. Such an idea was conceived by H. Minkowski for the first time, who formulated a 4-D space with 3 space co-ordinates (x, y, z) and one time co-ordinate (t). Such a space is named after him as Minkowski space. Such a space is indicated as (x_1, x_2, x_3, x_4), where $x_1 = x$, $x_2 = y$, $x_3 = z$ are the 3-cartesian co-ordinate and the fourth one $x_4 = ict$ ($i = \sqrt{-1}$, c, vel. of light) as the time co-ordinate. This space is 4 dimensional x_μ (with $\mu = 1, 2, 3, 4$) space / world space / Minkowski space / four space. A position vector in this space is given by

$$s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad \text{---(2)}$$

The Lorentz transformation equations are so stated it requires that a four vector remains invariant i.e.

$$s^2 = x'^2_\mu = x^2_\mu$$

$$= x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad \text{---(3)}$$

The Lorentz transformation equations are thus a set of orthogonal transformation in 4-D space.

Lorentz transformation eqn in 4-D space:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} r & 0 & 0 & i\beta r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta r & 0 & 0 & r \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{where } \beta = v/c,\ r = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} \longrightarrow (4)$$

$\Rightarrow$ $x_\mu' = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu$, $a_{\mu\nu}$ are the elements of 4×4 square matrix and x_ν — " — of (1×4) column matrix.

The inverse transformations are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} r & 0 & 0 & -i\beta r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta r & 0 & 0 & r \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} \longrightarrow (5)$$

or $x_\mu = \sum_{\nu=1}^{4} a_{\nu\mu} x_\nu'$ *

$$* \sum_\nu a_{\nu\mu} x_\nu' = \sum_\nu a_{\nu\mu} \sum_\lambda a_{\nu\lambda} x_\lambda = \sum_\lambda \sum_\nu a_{\nu\mu} a_{\nu\lambda} x_\lambda = \sum_\lambda \delta_{\mu\lambda} x_\lambda = x_\mu$$

As both x_μ and x_μ' satisfy eqn (6)

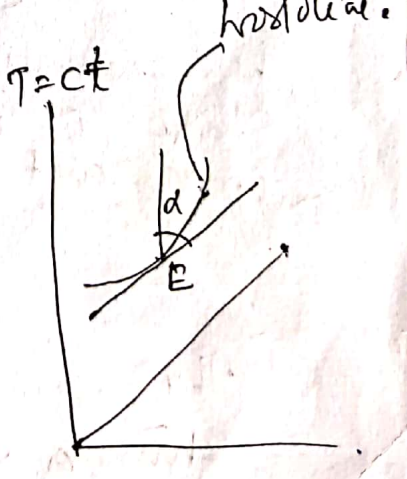
i.e. $\sum_{\mu=1}^{4} x_\mu'^2 = \sum_{\mu=1}^{4} x_\mu^2 \longrightarrow (6)$

World point and World line:

In the Minkowski space, a physical event is described by a point with four co-ordinate $(x_1, x_2, x_3, x_4 = ict)$. This point is called world point.

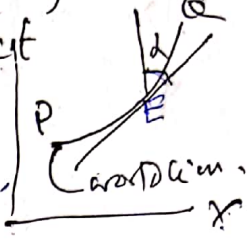
The motion of a particle corresponds to a line called world line.

Ex- for rectilinear motion, the line is a straight line.



The relative position of one event wrt. another is represented by a line element.

Consider a space time coordinate system, with one space axis x-axis and one time axis perpendicular to x axis by $T = ct$ (ignore y & z axis being irrelevant).



The Lorentz transformation for x & t are

$$x' = (x - vt)\gamma \text{ or } x' = (x - \beta T)\gamma \quad (7)$$

$$t' = \gamma\left(t - vx/c^2\right) \text{ or } T' = \gamma(T - \beta x) \quad (8)$$

$\beta = v/c$

$\beta T = \frac{v}{c} T = \frac{v}{c} \cdot ct = vt$

The L.T. eqn for space & time posses Symmetry. (C in (7) & (8), they have same form).

The motion is represented by the world line PQ.

The inclination α of the tangent to the world line at point E is

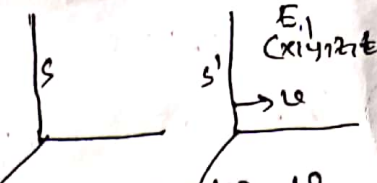
$$\tan\alpha = \frac{dx}{dT} = \frac{dx}{cdt} = \frac{v}{c} \quad (9)$$

$v < c$ for material particle & $\Rightarrow \alpha < 45^\circ$.

i) If v is const, $\tan\alpha$ is const, the world line is a st. line.

ii) for $v = c$ $\alpha = 45^\circ$ & world line ~~makes~~ for light signal is a st. line making 45°.

Space - Time interval:



S and S' are two inertial frames in relative motion with S' moving with v

$E_1(x_1, y_1, z_1, t_1)$ and $E_2(x_2, y_2, z_2, t_2)$ be two events measured wrt. S.

S_{12} the interval between these two events, the

$$S_{12}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - (ict_2 - ict_1)^2$$

$$= (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2 \quad \text{---(10)}$$

wrt to S' frame the interval appears

$$S_{12}'^2 = (x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2 - c^2(t_2'-t_1')^2 \quad \text{---(11)}$$

(10) and (11) represent the measure of separation between two events in Minkowski space.

making use of the Lorentz transformation equations (4) it can be shown that (one can verify)

$$S_{12}'^2 = S_{12}^2 \quad \text{---(12)}$$

So the interval is invariant under Lorentz transformation from one inertial frame to another.

However the time interval $(t_2'-t_1')^2 \neq (t_2-t_1)^2$ and also the space intervals $(x_2'-x_1')^2 + (y_2'-y_1')^2 + (z_2'-z_1')^2 \neq (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$, even though they satisfy (12).

This implies that space & time intervals are not invariant separately in special theory of relativity.

The interval S_{12} is independent of frame of reference and indicate the separation in Minkowski space.

Space and time has no independent significance but occurs as constituent of space-time together.

So a 3-D space 1-D time merge to form a 4-D space time continium.

Space like Interval:

We have $S_{12}^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 - c^2(t_2-t_1)^2$

$= r_{12}^2 - c^2(t_2-t_1)^2$

If $r_{12}^2 > c^2(t_2-t_1)^2$ then $S_{12}^2 > 0$ and the interval is called space like.

In such case we may not find a frame in which the two events may take place at same plane.

If so, $r_{12}^2 = 0$ and S_{12}^2 will be -ve.

So in space like interval we may find an inertial frame in which two events occur simultaneously ($t_2 = t_1$ i.e. $t_2 - t_1 = 0$)

So r_{12}^2 will have an appropriate value to give the same value of S_{12}^2 i.e.

$$S_{12}^{\prime 2} = r_{12}^{\prime 2} - c^2(t_2-t_1)^2 = r_{12}^{\prime 2} = S_{12}^{\prime 2}.$$

In some frame one event may occur earlier i.e. $t_1 < t_2$, in another the other may occur earlier $t_1 > t_2$ and in yet another they may occur simultaneously ($t_2 = t_1$)

So past, present and future can't be defined relative to these events.

So for space like interval the two events have no causal relationship. [For a causal relationship (cause & effect) between two events, one must occur after another. for ex, a bullet fired from a gun can't reach a target before being fired].

It may be seen that $r_{12}^2 > c^2(t_2-t_1)^2$ or $r_{12} > t_2 - t_1$ for space like interval

$\Rightarrow$ For space like interval, the time separation between the two events is less than the time taken by light covering the distance between them.

Time like Interval:

We have $s_{12}^2 = r_{12}^2 - c^2(t_2-t_1)^2$

If $c^2(t_2-t_1)^2 > r_{12}^2$, then $s_{12}^2 < 0$ i.e. -ve

The interval is imaginary & called time like interval. In such case the two events are not simultaneous in any inertial frame since $t_2 - t_1 \neq 0$.

Since $c^2(t_2-t_1)^2 > r_{12}^2$

$$\text{or } t_2 - t_1 > \frac{r_{12}}{c}$$

⇒ In time like interval:

The time separation between two events is more than the time taken by light to cover their distance of separation.

Light like interval:

If $s_{12} = 0$ the interval is said to be light like / Singular

So $r_{12}^2 - c^2(t_2-t_1)^2 = 0$

$$\Rightarrow r_{12}^2 = c^2(t_2-t_1)^2$$

$$\text{or } (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = c^2(t_2-t_1)^2$$

World region and light cone:

In Minkowski space the interval between two events $E_1(x_1, y_1, z_1, t_1)$ & $E_2(x_2, y_2, z_2, t_2)$ is given by $S_{12}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2$

or $S_{12}^2 = r_{12}^2 - c^2(t_2-t_1)^2$

This interval is Lorentz invariant $\left(S_{12}^2 = S_{12}'^2\right)$

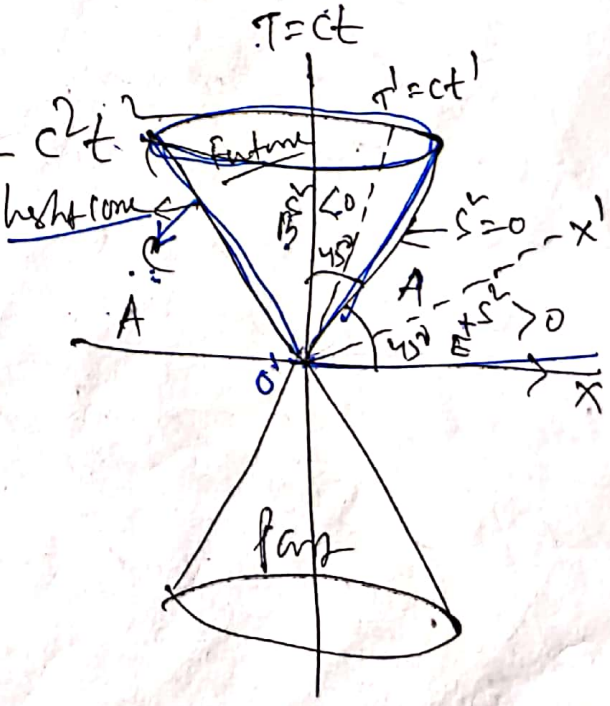
The interval is spacelike, timelike or lightlike depending on $S_{12}^2 >, <, = 0$ i.e. +ve, -ve or zero.

The characteristic of this interval is explained in a definite region in 4-D space called world region.

Consider that one event is at origin $O'(0,0,0,0)$ other event at $E(x, y, z, ict)$

Then the interval

$$S^2 = x^2 + y^2 + z^2 - c^2t^2$$



Define

(i) Region A $S^2 > 0$

(ii) Region B $S^2 < 0$

(iii) Sheet C $S^2 = 0$.

The sheet 'c' cone with vertex (4-1) minkowski space in region 1 & 3 is given by

$s^2 = 0$ or $x^2+y^2+z^2 = (ct)^2$

or $\sqrt{x^2+y^2+z^2} = ct$ or $r = ct$.

In 3 → $x^2+y^2 = z^2$ represent surface of a cone with apex at origin and axis along z axis with semi vertical angle $\pi/4 = 45^\circ$.

$\Rightarrow x^2+y^2+z^2 = c^2t^2$ (*)

In Analogy $s^2 = 0$ i.e. sheet c represent a cone with apex at origin and axis along time axis $T = ct$ & semi vertical angle $\pi/4$.

The value of $s^2 = 0$ on this surface of the cone.

So it is called a null cone.

So * represent all those world point which are sooner/later reached by light signal originally at 'o'. This cone is called light cone.

For region A → $s^2 > 0$ (spacelike), for B $s^2 < 0$ for sheet C $s^2 = 0$ (lightlike intervals) (timelike).

The upper cone refers to absolute future

Lower cone to absolute past.

relative to 'o'

Four Vector — 4-dimensional Minkowski space

Components transform as per Lorentz transformation.

An event in 4-D space is represented by world point (x_1, x_2, x_3, x_4).

The L.T. from S frame to S' frame in this space is given by

$$\beta = v/c \qquad \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}} = \gamma$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{(1)}$$

or $$x_\mu' = \sum_{\nu=1}^{4} A_{\mu\nu} x_\nu$$

Further $$x_\mu'^2 = x_\mu^2 \qquad \text{(2)}$$

The pos^n vector of the world point is

$$x_\mu = (x_1, x_2, x_3, x_4) = (\vec{r}, ict)$$

in IR $(x_1 x_2 x_3)$ or (x, y, z) represent pos^n vector $\vec{r}$ (space part)

& $x_4 = ict = iT$ is the time part.

In general a four vector is represented by

$$A_\mu = A_1, A_2, A_3, A_4 : (\vec{A}, iA_t)$$

$\vec{A}$ — space comp., iA_t — time comp.

The comp. transformation of 4 forces is from

$$\begin{pmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \quad (3)$$

or $$A_\mu' = \sum_{\nu=1}^{4} a_{\mu\nu} A_\nu \quad (4)$$

$$\begin{aligned} A_1' &= \gamma A_1 + i\beta\gamma A_4 \\ A_2' &= A_2 \\ A_3' &= A_3 \\ A_4' &= -i\beta\gamma A_1 + \gamma A_4 \end{aligned} \quad (5)$$

From this $A_\mu'^2 = A_\mu^2$ or $A_\mu' A_\mu' = A_\mu A_\mu$ (6)

Now $A_\mu^2 = A_1^2 + A_2^2 + A_3^2 + A_4^2$

$= A_1^2 + A_2^2 + A_3^2 - A_t^2 \quad (A_4 = iA_t)$

For two four vectors A_μ & B_μ.

$A_\mu B_\mu = A_1 B_1 + A_2 B_2 + A_3 B_3 + A_4 B_4$

$= A_1 B_1 + A_2 B_2 + A_3 B_3 - A_t B_t$

From this $A_\mu B_\mu = A_\mu' B_\mu'$ (invariant under L.T eq)

For 4th vector X_μ:

$$X_\mu = (x_1, x_2, x_3, x_4)$$
$$= (x, y, z, ict) = (\vec{r}, ict)$$

For velocity $u_\mu = \frac{dx_\mu}{d\tau}$.

$$u_1 = \frac{dx_1}{d\tau} = \frac{dx}{dt}\cdot\frac{dt}{d\tau} = u_x \cdot \frac{1}{\sqrt{1-u^2/c^2}}$$

Similarly $u_2 = u_y \cdot \frac{1}{\sqrt{1-u^2/c^2}}$

$$u_3 = u_z \cdot \frac{1}{\sqrt{1-u^2/c^2}}$$

$$u_4 = \frac{dx_4}{dt}\cdot\frac{dt}{d\tau} = ic \cdot \frac{1}{\sqrt{1-u^2/c^2}}$$

$$\therefore u_\mu = (u_1, u_2, u_3, u_4) = \left(\frac{u_x}{\sqrt{1-u^2/c^2}}, \frac{u_y}{\sqrt{1-u^2/c^2}}, \frac{u_z}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}}\right)$$

$$u_\mu = \left(\frac{\vec{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}}\right), \text{ where } \vec{u} = \frac{d\vec{r}}{dt}$$

This is 3-D velocity vector.

Now $u_\mu u_\mu = \frac{u^2}{1-u^2/c^2} - \frac{c^2}{1-u^2/c^2} = -c^2$

and is Lorentz invariant

For momentum $(p_\mu = p_1, p_2, p_3, p_4)$

$$p_1 = m_0 u_1 = \frac{m_0 u_x}{\sqrt{1-u^2/c^2}} = m u_x = p_x$$

$$p_2 = m_0 u_2 = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = m u_y = p_y$$

$$p_3 = m_0 u_3 = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}} = m u_z = p_z$$

$$p_4 = m_0 u_4 = \frac{m_0 ic}{\sqrt{1-u^2/c^2}} = imc = iE/c \quad (E = mc^2)$$

$\because m$

$$\therefore p_\mu = (p_1, p_2, p_3, p_4)$$

$$= (p_x, p_y, p_z, imc) = (\vec{p}, iE/c)$$

Now $p_\mu p_\mu = p^2 - E^2/c^2 = \frac{p^2c^2 - E^2}{c^2} = \frac{E^2 - p^2c^2}{c^2}$

$$= -m_0^2 c^2$$

$E/c = mc^2 \cdot c$

Four Acceleration:

$$a_1 = \frac{du_1}{d\tau} = \frac{du_1}{dt}\cdot\frac{dt}{d\tau} = \frac{du_1}{dt}\cdot\frac{1}{\sqrt{1-u^2/c^2}}$$

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$= \frac{1}{\sqrt{1-u^2/c^2}}\left[\frac{d}{dt}\left(\frac{u_x}{\sqrt{1-u^2/c^2}}\right)\right]$$

$$= \frac{1}{\sqrt{1-u^2/c^2}}\left[\frac{\sqrt{1-u^2/c^2}\cdot\frac{du_x}{dt} - u_x\cdot\frac{d}{dt}\left(\sqrt{1-u^2/c^2}\right)}{\left(\sqrt{1-u^2/c^2}\right)^2}\right]$$

$$= \frac{1}{\sqrt{1-u^2/c^2}}\left(\frac{\sqrt{1-u^2/c^2}\,\dot{u}_x - u_x\cdot\frac{d}{d(1-u^2/c^2)}\left(\sqrt{1-u^2/c^2}\right)\cdot\frac{d(1-u^2/c^2)}{dt}}{\left(\sqrt{1-u^2/c^2}\right)^2}\right)$$

$$= \frac{1}{\sqrt{1-u^2/c^2}}\cdot\frac{\sqrt{1-u^2/c^2}\cdot\dot{u}_x}{\left(\sqrt{1-u^2/c^2}\right)^2} - \frac{u_x}{\left(\sqrt{1-u^2/c^2}\right)^2}\cdot\frac{1}{2}\left(1-u^2/c^2\right)^{-1/2}\cdot\frac{-2u\cdot\dot{u}}{c^2}$$

$$= \frac{\dot{u}_x}{\left(\sqrt{1-u^2/c^2}\right)^2} + \frac{u_x\cdot u\cdot\dot{u}}{c^2\left(\sqrt{1-u^2/c^2}\right)^4}$$

$$a_1 = \frac{\dot{u}_x}{1-u^2/c^2} + \frac{u_x(\vec{u}\cdot\dot{\vec{u}})}{c^2(1-u^2/c^2)^2}$$

$u\dot{u} = u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z = \vec{u}\cdot\dot{\vec{u}}$

Similarly $a_2 = \frac{\dot{u}_y}{1-u^2/c^2} + \frac{u_y(\vec{u}\cdot\dot{\vec{u}})}{c^2(1-u^2/c^2)^2}$

$$a_3 = \frac{\dot{u}_z}{1-u^2/c^2} + \frac{u_z(\vec{u}\cdot\dot{\vec{u}})}{c^2(1-u^2/c^2)}$$

$$a_4 = \frac{du_4}{d\tau} = \frac{du_4}{dt}\cdot\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{ic}{\sqrt{1-u^2/c^2}}\right)\cdot\frac{1}{\sqrt{1-u^2/c^2}}$$

$$a_4 = \frac{i(\vec{u}\cdot\vec{a})}{c(1-u^2/c^2)^2}$$

So $$a_\mu = \frac{\vec{a}}{1-u^2/c^2} + \frac{\vec{u}(\vec{u}\cdot\vec{a})}{c^2(1-u^2/c^2)};\ \frac{i(\vec{u}\cdot\vec{a})}{c(1-u^2/a^2)^2}$$

Four Force:

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{d}{d\tau}(m_0 u_\mu) = m_0\frac{du_\mu}{d\tau} = m_0\frac{d^2x_\mu}{d\tau^2}$$

Called Minkowski Force eq^n.

In non relativistic limit $u \ll c$, the 3-D component

$$F_u = \frac{dp_u}{dt} = m_0\frac{d^2x_u}{dt^2} \rightarrow \text{Classical Newton Eq}^n.$$

The four force components are as follows:

$$F_1 = \frac{dp_1}{d\tau} = \frac{dp_1}{dt}\cdot\frac{dt}{d\tau} = \frac{dp_x}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{F_x}{\sqrt{1-u^2/c^2}}$$

Similarly $$F_2 = \frac{F_y}{\sqrt{1-u^2/c^2}},$$

$$F_3 = \frac{F_z}{\sqrt{1-u^2/c^2}}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow F_4 = \frac{dp_4}{d\tau} = \frac{dp_4}{dt}\frac{dt}{d\tau}\ \frac{d}{dt}(iE/c)\cdot\frac{1}{\sqrt{1-u^2/c^2}} = \frac{i\gamma}{c}\left(\frac{dE}{dt}\right)$$

Thus $$F_\mu = \left(\frac{\vec{F}}{\sqrt{1-u^2/c^2}}, \frac{i\gamma}{c}\frac{dE}{dt}\right)$$

with $\frac{dE}{dt} = \vec{F}.\vec{u}$ $\quad$ (Power $= \vec{F}.\vec{v}$)

$$F_\mu = \left(\frac{\vec{F}}{\sqrt{1-u^2/c^2}}, \frac{i(\vec{F}.\vec{u})}{c\sqrt{1-u^2/c^2}} \right).$$

Conservation of four momentum:

The four momentum is given by

$$P_\mu = \left(\vec{p}, iE/c\right) \text{ or } \left(\vec{p}, imc\right)$$

The space part is the linear momentum & time part contains energy E.

In a 2 particle elastic collision, the conservation of four momentum is stated as

$$P_\mu^1(B) + P_\mu^2(B) = P_\mu^1(A) + P_\mu^2(A)$$

A — After coll^n
B — Before coll^n.

or equivalently $\vec{p}_1(B) + \vec{p}_2(B) = p_1(A) + p_2(A)$

and $i\frac{E_1(B)}{c} + i\frac{E_2(B)}{c} = i\frac{E_1(A)}{c} + i\frac{E_2(A)}{c}$.

or $E_1(B) + E_2(B) = E_1(A) + E_2(A)$

Ex. Decay of unstable particle:

It is observed that heavy nuclei and unstable particles like π mesons decay spontaneously. We apply principle of conservation of momentum to explain such decay.

Consider a nucleus with rest mass = m_0

let it split into 2 comp with rest mass m_1 & m_2

For mom. of nucleus $p_\mu = (\vec{p}, iE/c)$

" of fragments $p_\mu^1 = (\vec{p}_1, iE_1/c)$

$p_\mu^2 = (\vec{p}_2, iE_2/c)$

By principle of conservation of mom.

$$p_\mu = p_\mu^1 + p_\mu^2 \quad \text{---(1)}$$

$$\Rightarrow \vec{p} = \vec{p}_1 + \vec{p}_2 \quad \text{---(2)} \quad \text{space comp.}$$

$$\& \quad E = E_1 + E_2 \quad \text{---(3)} \quad \text{Time comp.}$$

If the nucleus is at rest before decay

then $p_\mu = (0, im_0c)$ ---4

So (2) becom $\vec{p}_1 + \vec{p}_2 = 0$ ---(5)

(3) becom $E_1 + E_2 = m_0c^2$ ---(6)

From (5) $E_1 = \dfrac{m_1c^2}{\sqrt{1 - v_1^2/c^2}}$, $E_2 = \dfrac{m_2c^2}{\sqrt{1 - v_2^2/c^2}}$ ---(7)

For this $E_1 > m_1c^2$ & $E_2 > m_2c^2$

$$\text{w} \quad E_1 + E_2 \text{ (0)} > (m_1 + m_2)c^2$$

From (8) $\quad m_0c^2 > (m_1 + m_2)c^2$

$$\Rightarrow m_0 > (m_1 + m_2) \quad \text{---(8)}$$

or Rest mass of decaying nucleus > greater than rest mass of the components.

From (5) $\quad \vec{p_1} + \vec{p_2} = 0$

$$= \vec{p_1} = -\vec{p_2} = q \text{ (Say)}$$

Rel. expression for E

$$E^2 = p_0^2c^2 + m_0^2c^4$$

this (E1) = (8)

$$E_1^2 = p_1^2c^2 + m_1^2c^4$$

$$\text{w} \quad E_1 = \sqrt{q^2c^2 + m_1^2c^4} \quad \text{---(a)}$$

$$\text{2} \quad E_2 = \sqrt{q^2c^2 + m_2^2c^4} \quad \text{---(b)}$$

Put (a) & (b) in (6)

$$\sqrt{q^2c^2 + m_1^2c^4} + \sqrt{q^2c^2 + m_2^2c^4} = m_0c^2$$

$$\Rightarrow m_0 = \sqrt{m_1^2 + q^2/c^2} + \sqrt{m_2^2 + q^2/c^2} \quad \text{---(9)}$$

If m_0, m_1, m_2 are known then, we can find a momentum q & energy E_1 & E_2.

The decay of $\pi^{\pm}$ meson is given by

$$m_{\pi}^{\pm} \Longrightarrow \mu^{\pm} + \nu \text{ (neutrino)}$$

If $q \rightarrow$ helm momentum of one of the resulting particle ($\mu^{\pm}$) then

from (9)

$$m_{\pi}^{\pm} = \sqrt{m_{\mu}^{\pm 2} + q^2/c^2} + \sqrt{m_{\nu}^2 + q^2/c^2}$$

$$m_{\pi}^{\pm} = \sqrt{m_{\mu}^{\pm 2} + q^2/c^2} + q/c \qquad m_{\nu} = 0.$$

(10)

for μ^{+} meson $|\vec{q}| = 29.80 \pm 0.04$ mev/c.

& $m_{\mu}^{+} = 105.65 \pm 0.05$ mev.

Using (10)

$$m_{\pi}^{+} = 139.58 \pm 0.05 \text{ mev.}$$

and Conforms with experimental results.